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A Basic Integral Equation associated with $\overline{\mathrm{H}}$ - Function as Kernel

Abstract

The aim of the present paper is to obtain a new basic integral equation associating with the $\overline{\mathrm{H}}$ - function. In this article, we shows that how the systematic use of the theory of the Mellin transform leads to a simple procedure by means of which this class of integral equation may be solved. The technique, which presupposes the existence of the Euler transform of the kernel as well as the Mellin transform of the Euler transform, is illustrated here by obtaining of the integral equation. Since our main integral equation has the potential for specialization , a large number of new and known integral equations can be easily be obtained from our basic integral equation by taking some special functions that follows as particular cases of the $\overline{\mathrm{H}}$ - function[5]

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Introduction

Integral equation of the type

$$\int_0^x k(x/y) f(y)\frac{dy}{y} = g(x),\ x \geq 0 \tag{1.1}$$

Where g is prescribed and f is unknown function to be determined, can be reduced to the form

$$\int_u^\infty k_1(t-u) f_1(u) dt = g_1(xu),\ u \geq 0 \tag{1.2}$$

If we make the substitutions x = e^{-au} , y = e^{-bt} , and write

$f(e^{-az}) = f_1(z)$, $g(e^{-bz}) = g_1(z)$ and $k(e^{-cz}) = k_1(z)$

Our object the present paper is to show how the systematic use of the theory of the Mellin transform leads to a simple procedure by means of which this class of integral equations may be solved. The technique, which presuppose the existence of the the Euler transform of the kernel as well as the Mellin transform of the Euler transform, is illustrated here by obtaining of the integral equation

$$\int_y^\infty (x-y)^{-\alpha}\, \overline{\mathrm{H}}^{M,N}_{P,Q}\left[zy^{\lambda}(x-y)^{\mu} \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right. \right] f(x)dx = g(x) \tag{1.3}$$

Where $y \geq 0$ and $\overline{\mathrm{H}}^{M,N}_{P,Q}[z]$ denotes the $\overline{\mathrm{H}}$ - function defined by Innayat Hussain [5].

Preliminary resultes

We begin by recalling the definition of the Mellin transform F(s) of function f(x) as

$$F(s) = M\{f(x); s\} = \int_0^\infty x^{s-1} f(x)dx \tag{2.1}$$

By the Mellin inversion theorem[12, pp. 246-247], we know that if the integral in (2.1) converges absolutely on the line Re(s) = τ, and if f(t) is of bounded variation in a neighborhood of the point t = x (x > 0), then

$$M^{-}\{F(s)\} = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{\tau-iT}^{\tau+iT} x^{-s}F(s)\,ds = \frac{1}{2}\{f(x+0)+f(x-0)\} \qquad (2.2)$$

Wherein the least member may be replaced by f(x) if f(t) is continuos at the point t = x.

By Parseval`s theorem for the Mellin transform, it is know that if M {f(x)} = F(s) and M{$\mathcal{g}$ (x)} = G(s), then {10, p. 43}

$$\int_0^{\infty} f(x)\mathcal{g}(x)\,dx = \frac{1}{2\pi i}\int_{\tau-iT}^{\tau+iT} F(s)\,G(1-s)\,ds, \qquad (2.3)$$

Which holds true under two alternative sets of conditions enumerated in theorem[11, p. 60]

Next, we recall the definition of $\bar{H}$ - function which was introduced by Innayat Hussian [5] and it has been put on a firm footing by the research papers of Buschman and Srivastava [1], Rathi[7], Gupta and Soni[4], Saxena and Gupta[8], Saxena, Ram and Kalla[9] and several other research workers.

The $\bar{H}$ - function will be defined and represented as follows [1]

$$\bar{H}^{M,N}_{P,Q}[z] = \bar{H}^{M,N}_{P,Q}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\bar{\phi}(\xi)\,z^{\xi}\,d\xi \qquad (2.4)$$

Where

$$\bar{\phi} = \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_j\xi)\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \qquad (2.5)$$

Which contains fractional powers of some of the Gamma functions? Here, and throughout the papers and $i = \sqrt{-1}$. a_j (j = 1,…,P) and b_j (j = 1,…,Q) are complex parameters,

$\alpha_j \ge 0$ (j = 1,..., P) and $\beta_j \ge 0$ (j = 1,..., Q) and

the exponents A_j (j = 1,…,N) and B_j (j = N+1,…,Q) can take an non integer values which we assume to be positive for standardization purpose. The contour in (2.4) is imaginary axis Re(ε) = 0. In exact form can be seen in the paper cited earlier. When all the exponents A_j and B_j assume value unity, the H-function reduces to the well known Fox's H-function [3]. Buschman and Srivastava [1] has proved that the integral on the right hand side of (2.4) is absolutely convergent when Ω > 0 and | arg z | < 1/2 π Ω, where

$$\Omega = \sum_{j=1}^{M}\beta_j + \sum_{j=1}^{N}A_j\alpha_j - \sum_{j=M+1}^{Q}B_j\beta_j - \sum_{j=N+1}^{P}\alpha_j > 0 \qquad (2.6)$$

The following behavior of $\bar{H}^{M,N}_{P,Q}[z]$ for small and large values of z as reduced by Saxena and Gupta [8, p. 870, eq. (2.3) and eq. (2.4)] will be required in the sequel.

$$\bar{H}^{M,N}_{P,Q}[z] = O(|z|^{\mathcal{g}}) \quad \text{for small } z,$$

$$\text{where } \mathcal{g} = \min_{1\le j M}\{Re(\frac{b_j}{\beta_j})\}$$

$$\bar{H}^{M,N}_{P,Q}[z] = o(|z|^{h}) \quad \text{for large } z,$$

$$\text{where } h = \max_{1\le j N}\{Re[A_j(\frac{a_j-1}{\alpha_j})]\}$$

and the conditions given by (2.6) are also satisfied.

From (2.4), it follows that

$$M^{-}\{\bar{H}^{M,N}_{P,Q}[zx];s\} = \frac{\prod_{j=1}^{M}\Gamma(b_j+s\beta_j)\prod_{j=1}^{N}\{\Gamma(1-a_j-s\alpha_j)\}^{A_j}}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j-s\beta_j)\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j+s\alpha_j)} \qquad (2.7)$$

$$-\min_{1\le j\le M} Re(b_j/\beta_j) < Re(s) < \min_{1\le j\le N} A_j\,Re(1-a_j/\alpha_j)$$

And the conditions given by (2.6) and (2.7) are satisfied.

On substituting the contour integral (2.4) for the $\bar{H}$ - function, if we invert the order of integration and evaluate the inner integral by using the familier formula[2, p. 10].

$$\int_0^x y^{\lambda-1}(x-y)^{\mu-1}\,dy = x^{\lambda+\mu-1}\beta(\lambda,\mu) \qquad (2.8)$$

Where Re(λ) > 0 and Re(μ)> 0, we shall obtain the Euler transform

$$\int_0^x y^{-\alpha}(x-y)^{\alpha-\beta-1}\,\bar{H}^{M,N}_{P,Q}\left[zy^{\lambda}(x\text{-}y)^{\mu}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]dy$$

$$\frac{\Gamma(\alpha-\beta)}{x^{\beta}}\bar{H}^{M,N+2}_{P+2,Q+1}\left[zx^{\lambda+\mu}\left|\begin{matrix}(\alpha,\lambda;1),(1-\alpha+\beta,\mu;1),\ (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(\beta,\lambda+\mu;1),\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right] \qquad (2.9)$$

Valid when Re(β) < Re(α) < 1 + $\min_{1\le j\le M} Re(b_j/\beta_j)$ (2.10)

Solution of the problem

In this section we describe briefly our formal procedure to obtain the solution of the integral equation (1.3).

On setting $g(y) = (1+y)^{-1} y^{\gamma+1} h(y)$, if we multiply both side of equation (1.3) by $y^{\alpha-\beta-1}$ and integrate with respect of y over (0, ∞), we find upon inversion of the order of integration and by a subsequent change of variable, that

$$\int_0^\infty h(y)\, y^{\alpha-\beta-1}(1+y)^{-1}\,dy$$

$$= \int_0^\infty f(x)dx \int_0^x t^{-\alpha}(x-t)^{\alpha-\beta-1}\, \overline{H}^{M,N}_{P,Q}\left[zt^{\lambda}(x-t)^{\mu}\right]dt$$

$$= \Gamma(\alpha-\beta)\int_0^\infty f(x)\, x^{-\beta}\, \overline{H}^{M,N+2}_{P+2,Q+1}\left[zx^{\lambda+\mu}\left|\begin{matrix}(\alpha,\lambda;1),(1-\alpha+\beta,\mu;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(\beta,\lambda+\mu;1),(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]dx$$

By means of formula (2.9), provided that (2.10) holds true. Now, if we apply Parseval`s relation (2.3) in conjunction with the fact that M{f(x)} = F(x) implies M{ x^s f(x)} = F(s + δ), we shall obtain:

$$\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} \overline{H}(s)\,\Gamma(\alpha-\beta+\gamma-s+1)\,\Gamma(s-\alpha+\beta-\gamma)\,ds$$

$$= \frac{\Gamma(\alpha-\beta)}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} F(s)\,\frac{\Gamma(s-\alpha-\beta)}{\Gamma(s)}$$

x

$$\frac{\prod_{j=1}^{M}\Gamma\{b_j+\beta_j(1-s-\beta)\}\prod_{j=1}^{N}\{\Gamma[1-a_j-A_j(1-s-\beta)]\}^{A_j}}{\prod_{j=M+1}^{Q}\{\Gamma[1-b_j-\beta_j(1-s-\beta)]\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j+A_j(1-s-\beta))}\,ds \qquad (3.1)$$

Where H(s) = M{h(x)} = $\int_0^\infty x^{s-\gamma-1}(1+x^{-1})\,g(x)\,dx$

(3.2)

Since, by hypothesis h(x) = , $x^{-\gamma}(1+x^{-1})\,g(x)$, γ being a suitable constant. The Gamma function occurring inside the left hand integral in (3.1) have no poles in the region:

Re (α-β+γ) < Re(s) = ν < Re(α – β + γ) + 1

(3.3)

And the Gamma function occurring inside the right-hand integral in (3.1), hane neither poles nor zeros in the region:

0, Re(α - β), . . . < Re(s) = σ < . . .

(3.4)

where in the ellipses (…) indicate the numbers involving the parameters $\alpha_1,\ldots,\alpha_p$ and $\beta_1,\ldots,\beta_q$.

Assuming that it is possible to determine a common value of ν and σ satisfying the inequalities in (3.3) and (3.4). We may formally write an obvious consequences of (3.1) in the form

$$F(s) = \overline{H}\,\frac{\Gamma(s)\,\Gamma(\alpha-\beta+\gamma-s+1)\,\Gamma(s-\alpha+\beta-\gamma)}{\Gamma(\alpha-\beta)\,\Gamma(s-\alpha+\beta)}$$

x

$$\frac{\prod_{j=M+1}^{Q}\Gamma\{1-b_j-\beta_j(1-s-\beta)\}^{B_j}\prod_{j=N+1}^{P}\Gamma[a_j+\alpha_j(1-s-\beta)]}{\prod_{j=1}^{M}\Gamma[b_j+\beta_j(1-s-\beta)]\prod_{j=1}^{N}\{\Gamma[1-a_j+\alpha_j(1-s-\beta)]\}^{A_j}} \qquad (3.6)$$

Which by an applied to the Mellin inversion formula (2.2) and the equation (3.2) and on inverting the order of integration gives us :

$$f(x) = \frac{1}{2\pi i\,\Gamma(\alpha-\beta)}\int_0^\infty y^{-\gamma-1}(1+y^{-1})\,g(y)\,dy$$

x

$$\frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty}\frac{\Gamma(s)\,\Gamma(\alpha-\beta+\gamma-s+1)\,\Gamma(s-\alpha+\beta-\gamma)}{\Gamma(s-\alpha+\beta)}$$

x

$$\frac{\prod_{j=M+1}^{Q}\Gamma\{1-b_j-\beta_j(1-s-\beta)\}^{B_j}\prod_{j=N+1}^{P}\Gamma[a_j+\alpha_j(1-s-\beta)]}{\prod_{j=1}^{M}\Gamma[b_j+\beta_j(1-s-\beta)]\prod_{j=1}^{N}\{\Gamma[1-a_j+\alpha_j(1-s-\beta)]\}^{A_j}}\left(\frac{x}{y}\right)^{-s} \qquad (3.7)$$

Provided that the integrals involved are absolutely convergent and the function involved satisfying the conditions of the Mellin inversion theorem.

The inner contour integral in (3.7) can be interpreted fairly easily in terms of the $\overline{H}$ -function defined by (2.4).

Theorem

If the inequalities in (2.10) holds true, $y^{\alpha-\beta-1}\, g(y) \in L(0,\infty)$ and $y^{-\gamma-1}(1+y^{-1})\,g(y) \in L(0,\infty)$, then the integral equation (1.3) has for its solution:

$$f(x) = \frac{1}{\Gamma(\alpha-\beta)}\int_0^\infty y^{-\gamma-1}(1+y^{-1})\, g(y)$$

x

$$\bar{H}^{P-N+1,\,Q-M+2}_{Q+2,\,P+2}\left[\frac{y}{x}\middle|\begin{matrix}(b_j+\beta_j(1-\beta),\beta_j;B_j)_{M+1,Q},(1+\alpha-\beta+\gamma,1;1),(0,1;1),(b_j+\beta_j(1-\beta);\beta_j)_{1,M}\\(a_j+\alpha_j(1-\beta),\alpha_j)_{N+1,P},(\alpha-\beta+\gamma+1,1),(1+\alpha-\beta,1;1),(a_j+\alpha_j(1-\beta),\alpha_j;A_j)_{1,N}\end{matrix}\right]dy \quad (3.8)$$

$$f(x) = \frac{1}{\Gamma(\alpha-\beta)}\int_0^\infty y^{-\gamma-1}(1+y^{-1})\, g(y)$$

x

$$\bar{H}^{Q-M+2,\,P-N+1}_{P+2,\,Q+2}\left[\frac{x}{y}\middle|\begin{matrix}(1-a_j-\alpha_j(1-\beta),\alpha_j)_{N+1,P},(\beta-\alpha-\gamma,1),(\beta-\alpha,1;1),(1-a_j-\alpha_j(1-\beta),\alpha_j;A_j)_{1,N}\\(1-b_j-\beta_j(1-\beta);\beta_j)_{1,M},(1,1;1),(\beta-\alpha+\gamma,1;1),(1-b_j-\beta_j(1-\beta),\beta_j;B_j)_{M+1,Q}\end{matrix}\right]dy \quad (3.9)$$

The integral equation (1.3) obtained herein is simple and basic in nature and is likely to prove extremely useful for research workers in diverse fields of engineering, physics and statistics. If we take M=1, $\lambda = 0$ in (1.3) we arrive a known result basic formula [6, pp.320, eq.(1.9)].

Aim of the Study

The aim of this is to find the solution of a of a new basic integral equation associated with $\overline{H}$ function as kernel. In this article, we use the method of Mellin integral transform and the Euler transform of the kernel

Conclusion

In this paper, we have develop a new technique for solution of a general integral equation associated with $\overline{H}$ function as kernel. We have thus defined and studied a new integral equation in this article. It has more flexibility due general arguments of $\overline{H}-\text{function}$. Several special cases can be obtained which are known and new.

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